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An Algebraic Approach to Quantum Field Theory and Commutation Relations for Energy Momentum in the Framework of General Relativity

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We present a consistent set of commutation relations (C.R.) for a quantum system immersed in a classical gravitational field. The gravity field is described by metric tensor $g_{ik}(x)$ and $g_{00}(x)$ with coordinate gauge $g_{i0}=0$. The Hamiltonian of the system is found to be a linear function of $[-g_{00}(x)]^{1/2}$. Its properties we define by C.R. avoiding explicit expression in terms of fields, as well as its splitting into free and interaction parts. In this way a consistent set of C.R., which are equally simple for a flat and curvilinear space, can be established. To stress the main idea of our approach, we consider the simple but still nontrivial example of a scalar electrodynamics immersed in a gravity field. The electromagnetic current operator we define by its C.R. and not explicitly. An interesting feature of this approach is that the Poisson equation follows from the consistency of the C.R. The C.R. for the energy and momentum operators of the system in a gravity field are established which generalize the usual Poincare group generators C.R. For example, we find $(i/hc^2)[H_{\sigma(x)}, H_{\sigma'(x)}] = P_{\sigma\nabla\sigma'-\sigma'\nabla\sigma}$, where $H_{\sigma(x)}$ is the Hamiltonian of the system, which is a linear functional of $\sigma(x)\equiv[-g_{00}(x)]^{1/2}$ and $P_{s(x)}$ represents the momentum-density operator [averaged with the classical function $s(x)$].

1. We start with the Maxwell equations in a gravity field with the metric $g_{ik}(x)$, $g_{00}(x)$, $g_{i0}=0$,

$$\frac{1}{c}\frac{\partial D}{\partial t} = \nabla\times H, \qquad \frac{1}{c}\frac{\partial B}{\partial t} = -\nabla\times E,$$

$$E = (-g_{00})^{1/2}D, \qquad H = (-g_{00})^{1/2}B$$

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Effective dielectric and magnetic susceptibilities $\varepsilon = \mu = 1/\sigma(x) \equiv 1/[-g_{00}(x)]^{1/2}$ and $\nabla \times \ldots$ is a general covariant differential operation in the metric $g_{ik}(x)$ (Landau and Lifshitz, 1977). We avoid the usual quantization procedure (an introduction of potentials A_μ, finding the solutions, calculation of the energy and quantizing appropriate amplitudes) as complicated and awkward in the presence of gravity field. Instead, we treat the problem in terms of observable fields. We introduce q-number $D_{a(x)}$; in the classical limit, the measurement of $D_{a(x)}$ gives the result $\int D(x)a(x)d^3x$. Therefore, D_a can be considered a linear q-functional of that classical function $a(x)$ which describes the properties of a measuring device. The commutation relation (C.R.) between the electromagnetic fields can be written now as follows:

$$\frac{1}{hc}[D_a + B_b, D_c + B_d] = \int e^{ikl}(a_i d_{l,k} - c_i b_{l,k})\, dx$$

The Maxwell equations have the form

$$\frac{1}{c}\frac{\partial D_a}{\partial t} = \left(\frac{i}{hc}[H, D_a]\right) = B_{\sigma\nabla\times a}$$

We observe that the Hamiltonian H is a linear functional of the argument

$$\sigma(x) \equiv [-g_{00}(x)]^{1/2}$$

Taking into account the presence of charges, we can formulate dynamics in presence of gravity field,

$$\frac{i}{hc}[H_\sigma, D_a + B_b] = B_{\sigma\nabla\times a} - J_{\sigma a} - D_{\sigma\nabla\times b}$$

This equation can be considered as a definition of the C.R. of the energy operator H_σ.

2. The consistency of the scheme requires the fulfillment of Jacobi identities (J.I.) like

$$[H_\sigma,[B_a, B_b]] = [[H_\sigma, B_a]B_b] - [[H_\sigma, B_b]B_a]$$

The following C.R.,

$$i[B_a, J_s] = 0, \qquad i[D_a, J_s] = e^2\Lambda_{as} + \int \nabla\cdot[s\times(\nabla\times a)]\, dx$$

where Λ is a q-functional depending on scalar function $(a\cdot s)$ guarantee the consistency of our scheme. The continuity equation which contains the electric charge operator ρ_λ and other C.R. are

$$i[H_\sigma, \rho_\lambda] = J_{\sigma\nabla\lambda}, [D_a, \rho_\lambda] = 0, \qquad i[\rho_\lambda, B_a] = \int \nabla\cdot(\nabla\lambda \times a)\, dx,$$

$$i[\rho_\lambda, J_s] = \Lambda_{s\nabla\lambda}$$

(All operations are general covariant!) We must stress that the definition of H_σ will be complete only if we add the C.R. of this operator with a charged field (see part 4 below).

3. Our next step is an investigation of properties of $i[H_\sigma, H_{\sigma'}]$. We can find this using the J.I. (in the following $h = c = 1$):

$$i^2\left[[H_\sigma, H_{\sigma'}]B_a\right] = B_{s^k(a_{i,k} - a_{k,i})}, \qquad s_k = \sigma'\sigma_{,k} - \sigma\sigma'_{,k}$$

This suggests that the commutator $i[H_\sigma, H_{\sigma'}] \equiv -P_s$ depends on the functional argument $s(x) = \sigma'\nabla\sigma - \sigma\nabla\sigma'$:

$$i[P_s, B_a + D_b] = B_{s\times(\nabla\times a)} + D_{s\times(\nabla\times b)} - e\rho_{sb}$$

(In deriving this equation we have assumed C.R. $i[H_\sigma, J_s] = M_{\sigma s} + \rho_{\sigma\nabla s}$, which will be proved in part 4.) The C.R. for $[H_\sigma, H_{\sigma'}]$ does not contain any field operators and may have broader range of applicability (consider a nonrelativistic particle movement in a strong static gravity field. In this case we can write Hamiltonian explicitly $H_\sigma = (1/2m)p_i g^{ik}(x)\sigma(x)p_k$ with an additive constant $A\sigma(x)$ and add the C.R. $(i/h)[p_i, x^k] = \delta_i^k$. To satisfy $(i/hc)[H_\sigma, H_{\sigma'}] = -P_s$, it is necessary to put $A = mc^2$!)

4. We consider now a charge scalar field "immersed" in the same gravity field. Let $\phi_{\tau(x)}$ be a q-functional scalar field, ϕ^+ be its conjugated field, π_α the generalized momentum with the following C.R.:

$$i\left[\pi_\alpha + \pi_{\alpha'}^+ + \phi_\tau + \phi_{\tau'}^+ \pi_\beta + \pi_{\beta'}^+ + \phi_\rho + \phi_{\rho'}^+\right] = \int (\alpha'\rho + \alpha\rho' - \beta'\tau - \beta\tau')\, dx$$

and dynamics follows from

$$i[H_\sigma, \phi_\tau + \pi_\alpha] = \pi_{\gamma^{-1}\sigma\tau} + \phi_{(\sigma\gamma\alpha_{,i}g^{ik})_{,k} - m^2\sigma\gamma\alpha} \qquad \left(\gamma \equiv [\det g_{ik}(x)]^{1/2}\right)$$

We have also $[\rho_\lambda, \phi_\tau + \pi_\alpha] = -e\phi_{\tau\lambda} - e\pi_{\alpha\lambda}$, $i[H_\sigma, \rho_\lambda] = J_{\sigma\lambda_{,i}}$,

$$i[\rho_\lambda, J_s] = \Lambda_{s^i\lambda_{,i}}, \qquad i[H_\sigma, J_s] = M_{\sigma s_i} + \rho_{\sigma(s^i\gamma)_{,i}\gamma^{-1}}$$

(a) We can approve the form of $i[H_\sigma, H_{\sigma'}] = -P_{\sigma'\nabla\sigma - \sigma\nabla\sigma'}$

$$i[P_s, \phi_\tau + \pi_\alpha] = \phi_{(s^i\tau)_{,i}} + \pi_{s^i\alpha_{,i}}$$

We have also $i[P_s, \rho_\lambda] = \rho_{s^i\lambda_{,i}}$

$$i[P_s, J_u] = J_{\gamma^{-1}(s^k u^i \gamma)_{,k} - u^k s^i_{,k}}$$

(b) For the commutator $[P_s, P_{s'}]$, we find

$$[P_s, P_{s'}] = P_{s^k s'^i_{,k} - s'^k s^i_{,k}} \qquad \text{and} \qquad \left[P_{s_1}\left[P_{s_2}, P_{s_3}\right]\right] + \text{C.P.} = 0$$

is fulfilled.

(c) The following expression can be proved for the commutator:

$$i[H_\sigma, P_s] = H_{\sigma\gamma^{-1}(\gamma s^i)_{,i} - s^i\sigma_{,i}} + Q_{\sigma(s_{i;k} + s_{k;i})} + G_{\sigma s^k}$$

with the self-ajoint $Q_{\tau_{ik}}$

$$i\left[Q_{\tau_{ik}}, \phi_\tau + \pi_\alpha\right] = \phi_{(\tau^{ik}\gamma\alpha_{,i})_{,k} - (\tau^n_n\gamma\alpha_{,i}g^{ik})_{,k} + m^2\tau^n_n\gamma\alpha}; \left[G_u, \phi + \pi\right] = 0$$

and also

$$i\left[Q_{\tau_{ik}}, D_a + B_b\right] = -B_{\tau_{in}\gamma^{-1}a_{k,j}e^{ijk}} - eJ_{\tau_{ik}a^i} + eJ_{\tau^n{}_n a_k} + D_{\tau_{ik}b^i}$$

$$i\left[G_{u_i}, D_a + B_b\right] = B_{(u_k\gamma^{-1}a_{i,n}e^{ikn})_{,j}} - D_{(u_k\gamma^{-1}b_{i,n}e^{ikn})_{,j}} - e\rho_{u_k\gamma^{-1}b_{i,n}e^{ikn}}$$

From all these C.R. we can derive appropriate uncertainty relations for measurements of a pair of observables (in the presence or absence of a gravity field. For example, consider $[P_\lambda, J_u]$ or $[P_\lambda, D_a]$.)

It is interesting to note that we cannot, in general, obtain a closed algebra for $H_\sigma, P_s, Q_{\tau_{ik}}$. Only if s_i is a Killing field, $s_{i;k} + s_{k;i} = 0$, is the algebra of H, P closed.

In the next section we show that for the flat space the full set of C.R. for Poincaré group can be obtained and therefore relativistic invariance of the scheme is guaranteed.

5. We consider linear dependence of $\sigma(x) = \sigma_0 + n \cdot x$

$$(i/hc^2)\left[H_{\sigma_0 + n\cdot x}, H_{\sigma'_0 + m\cdot x}\right] = P_{\sigma_0 m - \sigma_0' n} + P_{(n\times m)\times x}$$

From our C.R. the interpretation of H and P follows: H_1 is the total energy, $H_{n \cdot x}$ is the booster operator, P_n is momentum along n, and $P_{n \times x}$ is the total angular momentum. Here we have 6 of 45 of Poincaré group C.R. From $i[P_s, P_{s'}] = P_{(s\nabla)s' - (s'\nabla)s}$ we get another 15 C.R. and the last 24 C.R. can be obtained from $i[H_\sigma, P_s]$.

If the gravity field cannot be removed by coordinate transformation, we get in a usual way uncertainty relations restricting precision of measurement of pairs of observables.

It is remarkable that the absence of magnetic charges and the Poisson equation follow from the conditions $[H_\sigma, G] = 0$. There is no need to postulate them in the beginning.

In conclusion, we summarize features of this approach to quantum field theory.

(a) The set of C.R. was proved to be consistent due to fulfillment of all J.I.

(b) The treatment of time and space is not symmetric; from $(i/hc^2)[H_\sigma, H_{\sigma'}] = P_{\sigma\nabla\sigma' - \sigma'\nabla\sigma}$ we can see that "square of energy is equal to momentum." However, relativistic invariance is guaranteed in the limit of a flat space.

(c) The general coordinate transformation is restricted to 3-space arbitrary coordinate transformations. The imposition of $g_{i0} = 0$ is always possible (there are four arbitrary functions in coordinate transformation) and it is justified because only due to $g_{i0} = 0$ it is possible to synchronize clocks in the space.

REFERENCES

Landau, L. D., and Lifshitz, E. M. (1977). *Classical Theory of Fields*. Pergamon Press, London; Addison-Wesley, Reading, Massachusetts.